3 First examples and properties

3.1 Construction of topologies and examples

Proposition 3.1. *Let V be a vector space and $\{p_\alpha\}_{\alpha \in A}$ a set of seminorms on V. For any finite subset $I \subseteq A$ and any $\epsilon > 0$ define*

$$U_{I,\epsilon} := \{x \in V : \forall \alpha \in I : p_\alpha(x) < \epsilon\}.$$

Then, the sets $U_{I,\epsilon}$ form the basis of the filter of neighborhoods of 0 in a topology on V that makes it into a tvs. Moreover, this topology is Hausdorff iff for any $x \in V \setminus \{0\}$ there exists $\alpha \in A$ such that $p_\alpha(x) > 0$.

Proof. **Exercise**. □

Theorem 3.2. *Let V be a tvs. Then, V is locally convex iff there exists a set of seminorms inducing its topology as in Proposition 3.1. Also, V is locally convex and pseudo-metrizable iff there exists a countable such set. Finally, V is semi-normable iff there exists a finite such set.*

Proof. **Exercise**. □

If V is a vector space over $\mathbb{K}$ and S is some set, then the set of maps $S \to V$ naturally forms a vector space over $\mathbb{K}$. This is probably the most important source of topological vector spaces in functional analysis. Usually, the spaces S and V carry additional structure (e.g. topologies) and the maps in question may be restricted, e.g. to be continuous etc. The topology given to this vector space of maps usually depends on these additional structures.

Example 3.3. Let S be a set and $F(S, \mathbb{K})$ be the set of functions on S with values in $\mathbb{K}$. Consider the set of seminorms $\{p_x\}_{x \in S}$ on $F(S, \mathbb{K})$ defined by $p_x(f) := |f(x)|$. This gives $F(S, \mathbb{K})$ the structure of a locally convex tvs. The topology defined in this way is also called the *topology of pointwise convergence*.

Exercise 4. Show that a sequence in $F(S, \mathbb{K})$ converges with respect to this topology iff it converges pointwise.

Example 3.4. Let S be a set and $B(S, \mathbb{K})$ be the set of bounded functions on S with values in $\mathbb{K}$. Then, $B(S, \mathbb{K})$ is a normed vector space with the supremum norm:

$$\|f\| := \sup_{x \in V} |f(x)| \quad \forall f \in B(S, \mathbb{K}).$$

The topology defined in this way is also called the *topology of uniform convergence*.

Exercise 5. Show that a sequence in $B(S,\mathbb{K})$ converges with respect to this topology iff it converges uniformly on all of S.

Exercise 6. (a) Show that on $B(S,\mathbb{K})$ the topology of uniform convergence is finer than the topology of pointwise convergence. (b) Under which circumstances are both topologies equal? (c) Under which circumstances is the topology of pointwise convergence metrizable?

Example 3.5. Let S be a topological space and $\mathfrak{K}$ the set of compact subsets of S. For $K \in \mathfrak{K}$ define on $\mathrm{C}(S,\mathbb{K})$ the seminorm

$$p_K(f) := \sup_{x\in K} |f(x)| \quad \forall f \in \mathrm{C}(S,\mathbb{K}).$$

The topology defined in this way on $\mathrm{C}(S,\mathbb{K})$ is called the *topology of compact convergence*.

Exercise 7. Show that a sequence in $\mathrm{C}(S,\mathbb{K})$ converges with respect to this topology iff it converges compactly, i.e., uniformly in any compact subset.

Exercise 8. (a) Show that on $\mathrm{C}(S,\mathbb{K})$ the topology of compact convergence is finer than the topology of pointwise convergence. (b) Show that on the space $\mathrm{C_b}(S,\mathbb{K})$ of bounded continuous maps the topology of uniform convergence is finer than the topology of compact convergence. (c) Give a sufficient condition for them to be equal.

Exercise 9. Let V be a vector space and $\{p_n\}_{n\in\mathbb{N}}$ a sequence of seminorms on V. Define the function $q : V \to \mathbb{R}^+_0$ via

$$q(x) := \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x)}{p_n(x)+1}.$$

(a) Show that q is a pseudo-semi-norm on V. (b) Show that the topology generated on V by q is the same as that generated by the sequence $\{p_n\}_{n\in\mathbb{N}}$.

Definition 3.6. Let S be a set, V a tvs. Let $\mathfrak{S}$ a set of subsets of S with the property that for X, Y in $\mathfrak{S}$ there exists $Z \in \mathfrak{S}$ such that $X \cup Y \subseteq Z$. Let $\mathcal{B}$ be a base of the filter of neighborhoods of 0 in V. Then, for $X \in \mathfrak{S}$ and $U \in \mathcal{B}$ the sets

$$M(X,U) := \{f \in F(S,V) : f(X) \subseteq U\}$$

define a base of the filter of neighborhoods of 0 for a translation invariant topology on $F(S,V)$. This is called the $\mathfrak{S}$-topology on $F(S,V)$.

Proposition 3.7. *Let S be a set, V a tvs and $\mathfrak{S} \subseteq \mathfrak{P}(S)$ as in Definition 3.6. Let $A \subseteq F(S,V)$ be a vector subspace. Then, A is a tvs with the the $\mathfrak{S}$-topology iff $f(X)$ is bounded for all $f \in A$ and $X \in \mathfrak{S}$.*

Proof. **Exercise.** □

Exercise 10. (a) Let S be a set and $\mathfrak{S}$ be the set of finite subsets of S. Show that the $\mathfrak{S}$-topology on $F(S,\mathbb{K})$ is the topology of pointwise convergence. (b) Let S be a topological space and $\mathfrak{K}$ the set of compact subsets of S. Show that the $\mathfrak{K}$-topology on $\mathrm{C}(S,\mathbb{K})$ is the topology of compact convergence. (c) Let S be a set and $\mathfrak{S}$ a set of subsets of S such that $S \in \mathfrak{S}$. Show that the $\mathfrak{S}$-topology on $F(S,\mathbb{K})$ is the topology of uniform convergence.

3.2 Completeness

In the absence of a metric we can use the vector space structure of a tvs to complement the information contained in the topology in order to define a Cauchy property which in turn will be used to define an associated notion of completeness.

Definition 3.8. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a tvs V is called a *Cauchy sequence* iff for every neighborhood U of 0 in V there is a number $N > 0$ such that for all $n, m \geq N : x_n - x_m \in U$.

Proposition 3.9. *Let V be a mvs with translation-invariant metric. Then, a sequence in V is Cauchy in the sense of Definition 3.8 iff it is Cauchy in the sense of Definition 1.49.*

Proof. Straightforward. □

This Proposition implies that there is no conflict with our previous definition of a Cauchy sequence in metric spaces if we restrict ourselves to translation-invariant metrics. Moreover, it implies that for this purpose it does not matter which metric we use, as long as it is translation-invariant. This latter condition is indeed essential.

Exercise 11. Give an example of an mvs with two compatible metrics d^1, d^2 and a sequence x, such that x is Cauchy with respect to d^1, but not with respect to d^2.

In the following, whenever we talk about a Cauchy sequence in a tvs (possibly with additional) structure, we mean a Cauchy sequence according to Definition 3.8.

For a topologically sensible notion of completeness, we need something more general than Cauchy sequences: Cauchy filters.

Definition 3.10. A filter $\mathcal{F}$ on a subset A of a tvs V is called a *Cauchy filter* iff for every neighborhood U of 0 in V there is an element $W \in \mathcal{F}$ such that $W - W \subseteq U$.

Proposition 3.11. *A sequence is Cauchy iff the associated filter is Cauchy.*

Proof. **Exercise.** □

Proposition 3.12. *A converging sequence is Cauchy. A converging filter is Cauchy.*

Proof. **Exercise.** □

Definition 3.13. A subset U of a tvs is called *complete* iff every Cauchy filter on U converges to a point in U. It is called *sequentially complete* iff every Cauchy sequence in U converges to a point in U.

Obviously, completeness implies sequential completeness, but not necessarily the other way round. Note that for an mvs with translation-invariant metric, completeness in the sense of metric spaces (Definition 1.52) is now called sequential completeness. However, we will see that in this context it is equivalent to completeness in the sense of the above definition.

Proposition 3.14. *Let V be a mvs. Then, V is complete (in the sense of tvs) iff it is sequentially complete.*

Proof. We have to show that sequential completeness implies completeness. (The opposite direction is obvious.) We use a translation-invariant metric on V. Suppose $\mathcal{F}$ is a Cauchy filter on V. That is, for any $\epsilon > 0$ there exists $U \in \mathcal{F}$ such that $U - U \subseteq B_\epsilon(0)$. Now, for each $n \in \mathbb{N}$ choose consecutively $U_n \in \mathcal{F}$ such that $U_n - U_n \subseteq B_{1/n}(0)$ and $U_n \subseteq U_{n-1}$ if $n > 1$ (possibly by using the intersection property). Thus, for every $N \in \mathbb{N}$ we have that for all $n, m \geq N : U_n - U_m \subseteq B_{1/N}(0)$. Now for each $n \in \mathbb{N}$ choose an element $x_n \in U_n$. These form a Cauchy sequence and by sequential completeness converge to a point $x \in V$. Given n observe that for all $y \in U_n$ $: d(y,x) \leq d(y,x_n) + d(x_n,x) < \frac{1}{n} + \frac{1}{n}$, hence $U_n \subseteq B_{2/n}(x)$ and thus $B_{2/n}(x) \in \mathcal{F}$. Since this is true for all $n \in \mathbb{N}$, $\mathcal{F}$ contains arbitrarily small neighborhoods of x and hence all of them, i.e., converges to x. □

Proposition 3.15. *(a) Let V be a Hausdorff tvs and A be a complete subset. Then A is closed. (b) Let V be a complete tvs and A be a closed subset. Then A is complete.*

Proof. **Exercise.** □

Exercise 12. Which of the topologies defined above are complete? Which become complete under additional assumptions on the space S?

3.3 Finite dimensional tvs

Theorem 3.16. *Let V be a Hausdorff tvs of dimension $n \in \mathbb{N}$. Then, any isomorphism of vector spaces from $\mathbb{K}^n$ to V is also an isomorphism of tvs. Moreover, any linear map from V to any tvs is continuous.*

Proof. We first show that any linear map from $\mathbb{K}^n$ to any tvs W is continuous. Define the map $g : \mathbb{K}^n \times W^n \to W$ given by

$$g((\lambda_1, \dots, \lambda_n), (v_1, \dots, v_n)) := \lambda_1 v_1 + \dots + \lambda_n v_n.$$

This map can be obtained by taking products and compositions of vector addition and scalar multiplication, which are continuous. Hence it is continuous. On the other hand, any linear map $f : \mathbb{K}^n \to W$ takes the form $f(\lambda_1, \dots, \lambda_n) = g((\lambda_1, \dots, \lambda_n), (v_1, \dots, v_n))$ for some fixed set of vectors $\{v_1, \dots, v_n\}$ in W and is thus continuous by Proposition 1.16.

We proceed to show that any linear map $V \to \mathbb{K}^n$ is continuous. We do this by induction in n starting with $n = 1$. For $n = 1$ any such non-zero map takes the form $g : \lambda e_1 \to \lambda$ for some $e_1 \in V \setminus \{0\}$. (If $g = 0$ continuity is trivial.) For $r > 0$ consider the element $re_1 \in V$. Since V is Hausdorff there exists an open neighborhood U of 0 in V that does not contain re_1. Moreover, we can choose U to be balanced. But then it is clear that $U \subseteq g^{-1}(B_r(0))$. That is, $g^{-1}(B_r(0))$ is a neighborhood of 0 in V. Since open balls centered at 0 form a base of neighborhoods of 0 in $\mathbb{K}$ this implies that the preimage of any neighborhood of 0 in $\mathbb{K}$ is a neighborhood of 0 in V. By Proposition 2.12.a this implies that g is continuous.

We now assume that we have proofed the statement in dimension $n-1$. Let V be a Hausdorff tvs of dimension n. Consider now some non-zero linear map $h : V \to \mathbb{K}$. We factorize h as $h = \tilde{h} \circ p$ into the projection $p : V \to V/\ker h$ and the linear map $\tilde{h} : V/\ker h \to \mathbb{K}$. $\ker h$ is a vector subspace of V of dimension $n-1$. In particular, it is a Hausdorff tvs and hence by assumption of the induction isomorphic as a tvs to $\mathbb{K}^{n-1}$. Thus, it is complete and by Proposition 3.15.a closed as a subspace of V. Therefore by Proposition 2.15 the quotient tvs $V/\ker h$ is Hausdorff. Since $V/\ker h$ is also one-dimensional it is isomorphic as a tvs to $\mathbb{K}$ as we have shown above. Thus, $\tilde{h}$ is continuous. Since the projection p is continuous by definition, the composition $h = \tilde{h} \circ p$ must be continuous. Hence, any linear map $V \to \mathbb{K}$ is continuous. But a linear map $V \to \mathbb{K}^n$ can be written as a composition of the continuous map $V \to V^n$ given by $v \mapsto (v, \ldots, v)$ with the product of n linear (and hence continuous) maps $V \to \mathbb{K}$. Thus, it must be continuous.

We have thus shown that for any n a Hausdorff tvs V of dimension n is isomorphic to $\mathbb{K}^n$ as a tvs via any vector space isomorphism. Thus, by the first part of the proof any linear map $V \to W$, where W is an arbitrary tvs must be continuous. $\square$

Definition 3.17. A topological space is called *locally compact* iff every point has a compact neighborhood.

Theorem 3.18 (Riesz)**.** *Let V be a Hausdorff tvs. Then, V is locally compact iff it is finite dimensional.*

Proof. If V is a finite dimensional Hausdorff tvs, then its is isomorphic to $\mathbb{K}^n$ for some n by Theorem 3.16. But closed balls around 0 are compact neighborhoods of 0 in $\mathbb{K}^n$, i.e., $\mathbb{K}^n$ is locally compact.

Now assume that V is a locally compact Hausdorff tvs. Let K be a compact and balanced neighborhood of 0. We can always find this since given a compact neighborhood by Proposition 2.9 we can find a balanced and closed subneighborhood which by Proposition 1.26 must then also be compact. Now let U be an open subneighborhood of $\frac{1}{2}K$. By compactness of K, there exists a finite set of points $\{x_1, \ldots, x_n\}$ such that $K \subseteq \bigcup_{i=1}^n (x_i + U)$. Let W be the finite dimensional subspace of V spanned by $\{x_1, \ldots, x_n\}$. By Theorem 3.16 W is isomorphic to $\mathbb{K}^m$ for some $m \in \mathbb{N}$ and hence complete and closed in V by Proposition 3.15. So by Proposition 2.15 the quotient space V/W is a Hausdorff tvs. Let $\pi : V \to V/W$ be the projection. Observe that, $K \subseteq W + U \subseteq W + \frac{1}{2}K$. Thus, $\pi(K) \subseteq \pi(\frac{1}{2}K)$, or equivalently $\pi(2K) \subseteq \pi(K)$. Iterating, we find $\pi(2^k K) \subseteq \pi(K)$ for all $k \in \mathbb{N}$ and hence $\pi(V) = \pi(K)$ since $V = \bigcup_{k=1}^\infty 2^k K$ as K is balanced. Since π is continuous $\pi(K) = \pi(V) = V/W$ is compact. But since V/W is Hausdorff any one dimensional subspace of it is isomorphic to $\mathbb{K}$ by Theorem 3.16 and hence complete and closed and would have to be compact. But $\mathbb{K}$ is not compact, so V/W cannot have any one-dimensional subspace, i.e., must have dimension zero. Thus, $W = V$ and V is finite dimensional. $\square$

<u>Exercise</u> 13. (a) Show that a finite dimensional tvs is always locally compact, even if it is not Hausdorff. (b) Give an example of an infinite dimensional tvs that is locally compact.

3.4 More on function spaces

Definition 3.19. A topological space S is called *completely regular* iff given a closed subset $C \subseteq S$ and a point $p \in S \setminus C$ there exists a continuous function $f : S \to [0, 1]$ such that $f(C) = \{0\}$ and $f(p) = 1$.

Definition 3.20. A topological space is called *normal* iff it is Hausdorff and if given two disjoint closed sets A and B there exist disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Lemma 3.21. *Let S be a normal topological space, U an open subset and C a closed subset such that $C \subseteq U$. Then, there exists an open subset U' and a closed subset C' such that $C \subseteq U' \subseteq C' \subseteq U$.*

Proof. **Exercise.** □

Theorem 3.22 (Uryson's Lemma)**.** *Let S be a normal topological space and A, B disjoint closed subsets. Then, there exists a continuous function $f : S \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof. Let $C_0 := A$ and $U_1 := S \setminus B$. Applying Lemma 3.21 we find an open subset $U_{1/2}$ and a closed subset $C_{1/2}$ such that

$$C_0 \subseteq U_{1/2} \subseteq C_{1/2} \subseteq U_1.$$

Performing the same operation on the pairs $C_0 \subseteq U_{1/2}$ and $C_{1/2} \subseteq U_1$ we obtain

$$C_0 \subseteq U_{1/4} \subseteq C_{1/4} \subseteq U_{1/2} \subseteq C_{1/2} \subseteq U_{3/4} \subseteq C_{3/4} \subseteq U_1.$$

We iterate this process, at step n replacing the pairs $C_{(k-1)/2^n} \subseteq U_{k/2^n}$ by $C_{(k-1)/2^n} \subseteq U_{(2k-1)/2^{n+1}} \subseteq C_{(2k-1)/2^{n+1}} \subseteq U_{k/2^n}$ for all $k \in \{1, \ldots, n\}$.

Now define

$$f(p) := \begin{cases} 1 & \text{if } p \in B \\ \inf\{x \in (0, 1] : p \in U_x\} & \text{if } p \notin B \end{cases}$$

Obviously $f(B) = \{1\}$ and also $f(A) = \{0\}$. To show that f is continuous it suffices to show that $f^{-1}([0, a))$ and $f^{-1}((b, 1])$ are continuous for $0 < a \leq 1$ and $0 \leq b < 1$. But,

$$f^{-1}([0, a)) = \bigcup_{x<a} U_x, \quad f^{-1}((b, 1]) = \bigcup_{x>b} (S \setminus C_x).$$

□

Corollary 3.23. *Every normal space is completely regular.*

Proposition 3.24. *Every metric space is normal.*

Proof. Let A, B be disjoint closed sets in the metric space S. For each $x \in A$ choose $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) \cap B = \emptyset$ and for each $y \in B$ choose $\epsilon_y > 0$ such that $B_{\epsilon_y}(y) \cap A = \emptyset$. Then, for any pair (x, y) with $x \in A$ and $y \in B$ we have $B_{\epsilon_x/2}(x) \cap B_{\epsilon_y/2}(y) = \emptyset$. Consider the open sets $U := \bigcup_{x\in A} B_{\epsilon_x/2}(x)$ and $V := \bigcup_{y\in B} B_{\epsilon_y/2}(y)$. Then, $U \cap V = \emptyset$, but $A \subseteq U$ and $B \subseteq V$. So S is normal. □

Definition 3.25. Let S, T be topological spaces and $F \subseteq \mathrm{C}(S, T)$. Then, F is called *equicontinuous* at $a \in S$ iff for all neighborhoods W of $f(a)$ in T there exists a neighborhood U of a in S such that $f(U) \subseteq W$ for all $f \in F$. Moreover, F is called *locally equicontinuous* iff F is equicontinuous for all $a \in S$.

<u>Exercise</u> 14. Let S be a topological space and $F \subseteq \mathrm{C}(S, \mathbb{K})$. (a) Show that F is bounded in $\mathrm{C}(S, \mathbb{K})$ with the topology of pointwise convergence iff for each $x \in S$ there exists $c > 0$ such that $|f(x)| < c$ for all $f \in F$. (b) Show that F is bounded in $\mathrm{C}(S, \mathbb{K})$ with the topology of compact convergence iff for each $K \subseteq S$ compact there exists $c > 0$ such that $|f(x)| < c$ for all $x \in K$ and for all $f \in F$.

Lemma 3.26. *Let S be a topological space and $F \subseteq \mathrm{C}(S, \mathbb{K})$ locally equicontinuous. Then, F is bounded with respect to the topology of pointwise convergence iff it is bounded with respect to the topology of compact convergence.*

Proof. **<u>Exercise</u>.** □

Lemma 3.27. *Let S be a topological space and $F \subseteq \mathrm{C}(S, \mathbb{K})$ locally equicontinuous. Then, the closures of F in the topology of pointwise convergence and in the topology of compact convergence are locally equicontinuous.*

Proof. **<u>Exercise</u>.** □

Proposition 3.28. *Let S be a topological space and $F \subseteq \mathrm{C}(S, \mathbb{K})$ locally equicontinuous. If F is closed then it is complete, both in the topology of pointwise convergence and in the topology of compact convergence.*

Proof. We first consider the topology of pointwise convergence. Let $\mathcal{F}$ be a Cauchy filter in F. For each $x \in S$ induce a filter $\mathcal{F}_x = e_x(\mathcal{F})$ on $\mathbb{K}$ through the evaluation map $e_x : \mathrm{C}(S, \mathbb{K}) \to \mathbb{K}$ given by $e_x(f) := f(x)$. Then each $\mathcal{F}_x$

is a Cauchy filter on $\mathbb{K}$ and thus convergent to a uniquely defined $g(x) \in \mathbb{K}$. This defines a function $g : S \to \mathbb{K}$. We proceed to show that g is continuous. Fix $a \in S$ and $\epsilon > 0$. By equicontinuity, there exists a neighborhood U of a such that $f(U) \subseteq B_\epsilon(0)$ for all $f \in F$ and hence $|f(x) - f(y)| < 2\epsilon$ for all $x, y \in U$ and $f \in F$. Fix $x, y \in U$. Then, there exists $f \in F$ such that $|f(x) - g(x)| < \epsilon$ and $|f(y) - g(y)| < \epsilon$. Hence

$$|g(x) - g(y)| \le |g(x) - f(x)| + |f(x) - f(y)| + |f(y) - g(y)| < 4\epsilon,$$

showing that g is continuous. Thus, $\mathcal{F}$ converges to g and $g \in F$ if F is closed.

We proceed to consider the topology of compact convergence. Let $\mathcal{F}$ be a Cauchy filter in F (now with respect to compact convergence). Then, $\mathcal{F}$ is also a Cauchy filter with respect to pointwise convergence and the previous part of the proof shows that there exists a function $g \in \mathrm{C}(S, \mathbb{K})$ to which $\mathcal{F}$ converges pointwise. But since $\mathcal{F}$ is Cauchy with respect to compact convergence it must convergence to g also compactly. Then, if F is closed we have $g \in F$ and F is complete. □

Definition 3.29. Let V be a tvs and $C \subseteq V$ a subset. Then, C is called *totally bounded* iff for each neighborhood U of 0 in V there exists a finite subset $F \subseteq C$ such that $C \subseteq F + U$.

Proposition 3.30. *Let V be a tvs and $C \subseteq V$ a totally bounded subset. Then, C is bounded.*

Proof. **Exercise.** □

Proposition 3.31. *Let C be a bounded subset of $\mathbb{K}^n$ with the standard topology. Then C is totally bounded.*

Proof. **Exercise.** □

Proposition 3.32. *Let V be a tvs and $C \subseteq V$ a compact subset. Then, C is complete and totally bounded.*

Proof. **Exercise.** □

Proposition 3.33. *Let V be a Hausdorff tvs and $C \subseteq V$ a subset. If C is totally bounded and complete then it is compact.*

If V is metrizable the above Proposition is simply a special case of Proposition 1.55. We will not provide the general proof here.

Definition 3.34. Let S be a topological space and $U \subseteq S$ a subset. Then, U is called *relatively compact* in S iff the closure of U in S is compact.

Theorem 3.35 (generalized Arzela-Ascoli). *Let S be a topological space. Let $F \subseteq \mathrm{C}(S, \mathbb{K})$ be locally equicontinuous and bounded in the topology of pointwise convergence. Then, F is relatively compact in $\mathrm{C}(S, \mathbb{K})$ with the topology of compact convergence.*

Proof. We consider the topology of compact convergence on $\mathrm{C}(S, \mathbb{K})$. By Lemma 3.26, F is bounded in this topology. The closure $\overline{F}$ of F is bounded by Proposition 2.10.c, equicontinuous by Lemma 3.27 and complete by Proposition 3.28. Due to Proposition 3.33 it suffices to show that $\overline{F}$ is totally bounded. Let U be a neighborhood of 0 in V. Then, there exists $K \subseteq S$ compact and $\epsilon > 0$ such that $U_{K,3\epsilon} \subseteq U$, where

$$U_{K,\delta} := \{f \in V : |f(x)| < \delta \ \forall x \in K\}.$$

By equicontinuity we can choose for each $a \in K$ a neighborhood W of a such that $|f(x) - f(a)| < \epsilon$ for all $x \in W$ and all $f \in \overline{F}$. By compactness of K there is a finite set of points $\{a_1, \ldots, a_n\}$ such that the associated neighborhoods $\{W_1, \ldots, W_n\}$ cover S. Now consider the continuous linear map $p : \mathrm{C}(S, \mathbb{K}) \to \mathbb{K}^n$ given by $p(f) := (f(a_1), \ldots, f(a_n))$. Since $\overline{F}$ is bounded, $p(\overline{F})$ is bounded in $\mathbb{K}^n$ (due to Proposition 2.12.b) and hence totally bounded (Proposition 3.31). Thus, there exists a finite subset $\{f_1, \ldots, f_m\} \subseteq \overline{F}$ such that $p(\overline{F})$ is covered by balls of radius ϵ centered at the points $p(f_1), \ldots, p(f_m)$. In particular, for any $f \in \overline{F}$ there is then $k \in \{1, \ldots, m\}$ such that $|f(a_i) - f_k(a_i)| < \epsilon$ for all $i \in \{1, \ldots, n\}$. Specifying also $x \in K$ there is $i \in \{1, \ldots, n\}$ such that $x \in W_i$. We obtain the estimate

$$|f(x) - f_k(x)| \le |f(x) - f(a_i)| + |f(a_i) - f_k(a_i)| + |f_k(a_i) - f_k(x)| < 3\epsilon.$$

Since $x \in K$ was arbitrary this implies $f \in f_k + U_{K,3\epsilon} \subseteq f_k + U$. We conclude that $\overline{F}$ is covered by the set $\{f_1, \ldots, f_m\} + U$. Since U was an arbitrary neighborhood of 0 this means that $\overline{F}$ is totally bounded. □

Proposition 3.36. *Let S be a locally compact space. Let $F \subseteq \mathrm{C}(S, \mathbb{K})$ be totally bounded in the topology of compact convergence. Then, F is equicontinuous.*

Proof. **Exercise.** □

3.5 The Hahn-Banach Theorem

Theorem 3.37 (Hahn-Banach). *Let V be a vector space over $\mathbb{K}$, p be a seminorm on V, $A \subseteq V$ a vector subspace. Let $f : A \to \mathbb{K}$ be a linear map such that $|f(x)| \leq p(x)$ for all $x \in A$. Then, there exists a linear map $\tilde{f} : V \to \mathbb{K}$, extending f (i.e., $\tilde{f}(x) = f(x)$ for all $x \in A$) and such that $|f(x)| \leq p(x)$ for all $x \in V$.*

Proof. We first consider the case $\mathbb{K} = \mathbb{R}$. Suppose that A is a proper subspace of V. Let $v \in V \setminus A$ and define B to be the subspace of V spanned by A and v. In a first step we show that there exists a linear map $\tilde{f} : B \to \mathbb{R}$ such that $\tilde{f}(x) = f(x)$ for all $x \in A$ and $|f(y)| \leq p(y)$ for all $y \in B$. Since any vector $y \in B$ can be uniquely written as $y = x + \lambda v$ for some $x \in A$ and some $\lambda \in \mathbb{R}$, we have $\tilde{f}(y) = f(x) + \lambda \tilde{f}(v)$, i.e, $\tilde{f}$ is completely determined by its value on v. For all $x, x' \in A$ we have

$$f(x) + f(x') = f(x + x') \leq p(x + x') \leq p(x - v) + p(x' + v)$$

and thus,

$$f(x) - p(x - v) \leq p(x' + v) - f(x').$$

In particular, defining a to be the supremum for $x \in A$ on the left and b to be the infimum for $y \in A$ on the right we get

$$a = \sup_{x \in A}\{f(x) - p(x - v)\} \leq \inf_{x' \in A}\{p(x' + v) - f(x')\} = b.$$

Now choose $c \in [a, b]$ arbitrary. We claim that by setting $\tilde{f}(v) := c$, $\tilde{f}$ is bounded by p as required. For $x \in A$ and $\lambda > 0$ we get

$$\tilde{f}(x + \lambda v) = \lambda \left(\tilde{f}\left(\lambda^{-1} x\right) + c \right) \leq \lambda\, p\left(\lambda^{-1} x + v\right) = p\left(x + \lambda v\right)$$
$$\tilde{f}(x - \lambda v) = \lambda \left(\tilde{f}\left(\lambda^{-1} x\right) - c \right) \leq \lambda\, p\left(\lambda^{-1} x - v\right) = p\left(x - \lambda v\right).$$

Thus, we get $\tilde{f}(x) \leq p(x)$ for all $x \in B$. Replacing x by $-x$ and using that $p(-x) = p(x)$ we obtain also $-\tilde{f}(x) \leq p(x)$ and thus $|\tilde{f}(x)| \leq p(x)$ as required.

We proceed to the second step of the proof, showing that the desired linear form $\tilde{f}$ exists on V. We will make use of Zorn's Lemma. Consider the set of pairs $(W, \tilde{f})$ of vector subspaces $A \subseteq W \subseteq V$ with linear forms $\tilde{f} : W \to \mathbb{R}$ that extend f and are bounded by p. These pairs are partially ordered by extension, i.e., $(W, \tilde{f}) \leq (W', \tilde{f}')$ iff $W \subseteq W$ and $\tilde{f}'|_W = \tilde{f}$. Moreover, for any totally ordered subset of pairs $\{(W_i, \tilde{f}_i)\}_{i \in I}$ there is an upper bound given by $(W_I, \tilde{f}_I)$ where $W_I := \bigcup_{i\in I} W_i$ and $\tilde{f}_I(x) := \tilde{f}_i(x)$ for $x \in W_i$. Thus, by Zorn's Lemma there exists a maximal pair $(W, \tilde{f})$. Since the first part of the proof has shown that for any proper vector subspace of V we can construct an extension, i.e., a pair that is strictly greater with respect to the ordering, we must have $W = V$. This concludes the proof in the case $\mathbb{K} = \mathbb{R}$.

We turn to the case $\mathbb{K} = \mathbb{C}$. Let $f_r(x) := \Re f(x)$ for all $x \in A$ be the real part of the linear form $f : A \to \mathbb{C}$. Since the complex vector spaces A and V are also real vector spaces and p reduces to a real seminorm, we can apply the real version of the proof to f_r to get a real linear map $\tilde{f}_r : V \to \mathbb{R}$ extending f_r and being bounded by p. We claim that $\tilde{f} : V \to \mathbb{C}$ given by

$$\tilde{f}(x) := \tilde{f}_r(x) - \mathrm{i}\tilde{f}_r(\mathrm{i}x) \quad \forall x \in V$$

is then a solution to the complex problem. We first verify that $\tilde{f}$ is complex linear. Let $x \in V$ and $\lambda \in \mathbb{C}$. Then, $\lambda = a + \mathrm{i}b$ with $a, b \in \mathbb{R}$ and

$$\begin{aligned}
\tilde{f}(\lambda x) &= a\tilde{f}(x) + b\tilde{f}(\mathrm{i}x)\\
&= a\tilde{f}_r(x) - a\mathrm{i}\tilde{f}_r(\mathrm{i}x) + b\tilde{f}_r(\mathrm{i}x) + b\mathrm{i}\tilde{f}_r(x)\\
&= (a + \mathrm{i}b)\left(\tilde{f}_r(x) - \mathrm{i}\tilde{f}_r(\mathrm{i}x)\right)\\
&= \lambda\tilde{f}(x).
\end{aligned}$$

We proceed to verify that $\tilde{f}(x) = f(x)$ for all $x \in A$. For all $x \in A$,

$$\tilde{f}(x) = \Re f(x) - \mathrm{i}\Re f(\mathrm{i}x) = \Re f(x) - \mathrm{i}\Re(\mathrm{i}f(x)) = \Re f(x) + \mathrm{i}\Im(f(x)) = f(x).$$

It remains to show that $\tilde{f}$ is bounded by p. Let $x \in V$. Choose $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\lambda\tilde{f}(x) \in \mathbb{R}$. Then,

$$\left|\tilde{f}(x)\right| = \left|\lambda\tilde{f}(x)\right| = \left|\tilde{f}(\lambda x)\right| = \left|\tilde{f}_r(\lambda x)\right| \le p(\lambda x) = p(x).$$

This completes the proof. □

Corollary 3.38. *Let V be a normed vector space, $c > 0$, $A \subseteq V$ a vector subspace and $f : A \to \mathbb{K}$ a linear form satisfying $|f(x)| \le c\|x\|$ for all $x \in A$. Then, there exists a linear form $\tilde{f} : V \to \mathbb{K}$ that coincides with f on A and satisfies $|\tilde{f}(x)| \le c\|x\|$ for all $x \in V$.*

Proof. Immediate. □

Theorem 3.39. *Let V be a locally convex tvs, $A \subseteq V$ a vector subspace and $f : A \to \mathbb{K}$ a continuous linear form. Then, there exists a continuous linear form $\tilde{f} : V \to \mathbb{K}$ that coincides with f on A.*

Proof. Since f is continuous on A, the set $U := \{x \in A : |f(x)| \le 1\}$ is a neighborhood of 0 in A. Since A carries the subset topology, there exists a neighborhood $\tilde{U}$ of 0 in V such that $\tilde{U} \cap A \subseteq U$. By local convexity, there exists a convex and balanced subneighborhood $W \subseteq \tilde{U}$ of 0 in V. The associated Minkowski functional

$$p(x) := \inf\{\lambda \in \mathbb{R}_0^+ : x \in \lambda W\}$$

is a seminorm on V and we have $|f(x)| \le p(x)$ for all $x \in A$. Thus, we may apply the Hahn-Banach Theorem 3.37 to obtain a linear form $\tilde{f} : V \to \mathbb{K}$ that coincides with f on the subspace A and is bounded by p. Since p is continuous this implies that $\tilde{f}$ is continuous. □

Corollary 3.40. *Let V be a locally convex Hausdorff tvs. Then,* $\mathrm{CL}(V, \mathbb{K})$ *separates points in V. That is, for any pair $x, y \in V$ such that $x \neq y$, there exists $f \in \mathrm{CL}(V, \mathbb{K})$ such that $f(x) \neq f(y)$.*

Proof. **Exercise.** □

3.6 More examples of function spaces

Definition 3.41. Let T be a locally compact space. A continuous function $f : T \to \mathbb{K}$ is said to *vanish at infinity* iff for any $\epsilon > 0$ the subset $\{x \in T : |f(x)| \ge \epsilon\}$ is compact in T. The set of such functions is denoted by $\mathrm{C}_0(T, \mathbb{K})$.

Exercise 15. Let T be a locally compact space. Show that $\mathrm{C}_0(T, \mathbb{K})$ is complete in the topology of uniform convergence, but not in general complete in the topology of compact convergence.

Definition 3.42. Let U be a non-empty open subset of $\mathbb{R}^n$. For a multi-index $l \in \mathbb{N}_0^n$ we denote the corresponding partial derivative of a function $f : \mathbb{R}^n \to \mathbb{K}$ by

$$D^l f := \frac{\partial^{l_1} \dots \partial^{l_n}}{\partial x_1^{l_1} \dots \partial x_n^{l_n}} f.$$

Let $k \in \mathbb{N}_0$. If all partial derivatives with $|l| := l_1 + \cdots + l_n \le k$ for a function f exist and are continuous, we say that f is k times continuously differentiable. We denote the vector space of k times *continuously differentiable* functions on U with values in $\mathbb{K}$ by $\mathrm{C}^k(U,\mathbb{K})$. We say a function $f: U \to \mathbb{K}$ is *infinitely differentiable* or *smooth* if it is k times continuously differentiable for any $k \in \mathbb{N}_0$. The corresponding vector space is denoted by $\mathrm{C}^\infty(U,\mathbb{K})$.

Definition 3.43. Let U be a non-empty open and bounded subset of $\mathbb{R}^n$ and $k \in \mathbb{N}_0$. We denote by $\mathrm{C}^k(\overline{U},\mathbb{K})$ the set of continuous functions $f: \overline{U} \to \mathbb{K}$ that are k times continuously differentiable on U, and such that any partial derivative $D^l f$ with $|l| \le k$ extends continuously to $\overline{U}$. Similarly, we denote by $\mathrm{C}^\infty(\overline{U},\mathbb{K})$ the set of continuous functions $f: \overline{U} \to \mathbb{K}$, smooth in U and such that any partial derivative extends continuously to $\overline{U}$.

Example 3.44. Let U be a non-empty open and bounded subset of $\mathbb{R}^n$. Let $l \in \mathbb{N}_0^n$ and define the seminorm $p_l : \mathrm{C}^k(\overline{U},\mathbb{K}) \to \mathbb{R}_0^+$ via

$$p_l(f) := \sup_{x \in \overline{U}} \left| \left(D^l f \right)(x) \right|$$

for $k \in \mathbb{N}_0$ with $k \ge |l|$ or for $k = \infty$. For any $k \in \mathbb{N}_0$ the set of seminorms $\{p_l : l \in \mathbb{N}_0^n, |l| \le k\}$ makes $\mathrm{C}^k(\overline{U},\mathbb{K})$ into a normable vector space. Similarly, the set of seminorms $\{p_l : l \in \mathbb{N}_0^n\}$ makes $\mathrm{C}^\infty(\overline{U},\mathbb{K})$ into a locally convex mvs.

Exercise 16. Let U be a non-empty open and bounded subset of $\mathbb{R}^n$. Show that $\mathrm{C}^\infty(\overline{U},\mathbb{K})$ with the topology defined above is complete, but not normable.

Definition 3.45. A topological space is called *σ-compact* iff it is locally compact and admits a covering by countably many compact subsets.

Definition 3.46. Let T be a topological space. A *compact exhaustion* of T is a sequence $\{U_i\}_{i\in\mathbb{N}}$ of open and relatively compact subsets such that $\overline{U_i} \subseteq U_{i+1}$ for all $i \in \mathbb{N}$ and $\bigcup_{i\in\mathbb{N}} U_i = T$.

Proposition 3.47. *A topological space admits a compact exhaustion iff it is σ-compact.*

Proof. Suppose the topological space T is σ-compact. Then there exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact subsets such that $\bigcup_{n\in N} K_n = T$. Since T is locally compact, every point possesses an open and relatively compact neighborhood. (Take an open subneighborhood of a compact neighborhood.) We cover K_1 by such open and relatively compact neighborhoods around every point. By compactness a finite subset of those already covers K_1. Their union, which we call U_1, is open and relatively compact. We proceed inductively. Suppose we have constructed the open and relatively compact set U_n. Consider the compact set $\overline{U_n} \cup K_{n+1}$. Covering it with open and relatively compact neighborhoods and taking the union of a finite subcover we obtain the open and relatively compact set U_{n+1}. It is then clear that the sequence $\{U_n\}_{n\in\mathbb{N}}$ obtained in this way provides a compact exhaustion of T since $\overline{U_i} \subseteq U_{i+1}$ for all $i \in \mathbb{N}$ and $T = \bigcup_{n\in\mathbb{N}} K_n \subseteq \bigcup_{n\in\mathbb{N}} U_n$.

Conversely, suppose T is a topological space and $\{U_n\}_{n\in\mathbb{N}}$ is a compact exhaustion of T. Then, the sequence $\{\overline{U_n}\}_{n\in\mathbb{N}}$ provides a countable covering of T by compact sets. Also, given $p \in T$ there exists $n \in \mathbb{N}$ such that $p \in U_n$. Then, the compact set $\overline{U_n}$ is a neighborhood of p. That is, T is locally compact. □

Proposition 3.48. *Let T be a topological space, $K \subseteq T$ a compact subset and $\{U_n\}_{n\in\mathbb{N}}$ a compact exhaustion of T. Then, there exists $n \in \mathbb{N}$ such that $K \subseteq U_n$.*

Proof. **Exercise.** □

Proposition 3.49. *Let T be a σ-compact space. Then, $\mathrm{C}(T,\mathbb{K})$ with the topology of compact convergence is metrizable.*

Proof. **Exercise.** □

Example 3.50. Let U be a non-empty open subset of $\mathbb{R}^n$ and $k \in \mathbb{N}_0 \cup \{\infty\}$. Let W be an open and bounded subset of $\mathbb{R}^n$ such that $\overline{W} \subseteq U$ and let $l \in \mathbb{N}_0^n$ such that $|l| \le k$. Define the seminorm $p_{\overline{W},l} : \mathrm{C}^k(U,\mathbb{K}) \to \mathbb{R}_0^+$ via

$$p_{\overline{W},l}(f) := \sup_{x\in\overline{W}} \left| \left(D^l f\right)(x) \right|.$$

The set of these seminorms makes $\mathrm{C}^k(U,\mathbb{K})$ into a locally convex tvs.

Exercise 17. Let $U \subseteq \mathbb{R}^n$ be non-empty and open and let $k \in \mathbb{N}_0 \cup \{\infty\}$. Show that $\mathrm{C}^k(U,\mathbb{K})$ is complete and metrizable, but not normable.

Exercise 18. Let $0 \le k < m \le \infty$. (a) Let $U \subset \mathbb{R}^n$ be non-empty, open and bounded. Show that the inclusion map $\mathrm{C}^m(\overline{U},\mathbb{K}) \to \mathrm{C}^k(\overline{U},\mathbb{K})$ is injective and continuous, but does not in general have closed image. (b) Let $U \subseteq \mathbb{R}^n$ be non-empty and open. Show that the inclusion map $\mathrm{C}^m(U,\mathbb{K}) \to \mathrm{C}^k(U,\mathbb{K})$ is injective and continuous, but is in general neither bounded nor has closed image.

Exercise 19. Let $U \subset \mathbb{R}^n$ be non-empty, open and bounded, let $k \in \mathbb{N}_0 \cup \{\infty\}$. Show that the inclusion map $\mathrm{C}^k(\overline{U}, \mathbb{K}) \to \mathrm{C}^k(U, \mathbb{K})$ is injective and continuous. Show also that its image is in general not closed.

Exercise 20. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. For $f \in \mathrm{C}^1(\mathbb{R}, \mathbb{K})$ consider the operator $D(f) := f'$. (a) Show that $D : \mathrm{C}^{k+1}([0,1], \mathbb{K}) \to \mathrm{C}^k([0,1], \mathbb{K})$ is continuous. (b) Show that $D : \mathrm{C}^{k+1}(\mathbb{R}, \mathbb{K}) \to \mathrm{C}^k(\mathbb{R}, \mathbb{K})$ is continuous.

Exercise 21. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. For $f \in \mathrm{C}(\mathbb{R}, \mathbb{K})$ consider the operator

$$(I(f))(y) := \int_0^y f(x)\,\mathrm{d}x.$$

(a) Show that $I : \mathrm{C}^k([0,1], \mathbb{K}) \to \mathrm{C}^{k+1}([0,1], \mathbb{K})$ is continuous. (b) Show that $I : \mathrm{C}^k(\mathbb{R}, \mathbb{K}) \to \mathrm{C}^{k+1}(\mathbb{R}, \mathbb{K})$ is continuous.

Definition 3.51. Let D be a non-empty, open and connected subset of $\mathbb{C}$. We denote by $\mathcal{O}(D)$ the vector space of holomorphic functions on D. If D is also bounded we denote by $\mathcal{O}(\overline{D})$ the vector space of complex continuous functions on $\overline{D}$ that are holomorphic in D.

Exercise 22. (a) Show that $\mathcal{O}(\overline{D})$ is complete with the topology of uniform convergence. (b) Show that $\mathcal{O}(D)$ is complete with the topology of compact convergence.

Theorem 3.52 (Montel)**.** *Let $D \subseteq \mathbb{C}$ be non-empty, open and connected and $F \subseteq \mathcal{O}(D)$. Then, the following are equivalent:*

1. *F is relatively compact.*
2. *F is totally bounded.*
3. *F is bounded.*

Proof. 1.⇒2. $\overline{F}$ is compact and hence totally bounded by Proposition 1.55. Since F is a subset of $\overline{F}$ it must also be totally bounded. 2.⇒3. This follows from Proposition 3.30. 3.⇒1. Since D is locally compact, it is easy to see that boundedness is equivalent to the following property: For each point $z \in D$ there exists a neighborhood $U \subseteq D$ and a constant $M > 0$ such that $|f(x)| \le M$ for all $x \in U$ and all $f \in F$. It can then be shown that F is locally equicontinuous [Notes on Complex Analysis, Theorem 4.31]. The Arzela-Ascoli Theorem 3.35 then ensures that F is relatively compact. $\square$

Definition 3.53. Let X be a measurable space, μ a measure on X and $p > 0$. Define

$$\mathcal{L}^p(X, \mu, \mathbb{K}) := \{f : X \to \mathbb{K} \text{ measurable} : |f|^p \text{ integrable}\}.$$

Also define

$$\mathcal{L}^\infty(X, \mu, \mathbb{K}) := \{f : X \to \mathbb{K} \text{ measurable} : |f| \text{ bounded almost everywhere}\}.$$

We recall the following facts from real analysis.

Example 3.54. The set $\mathcal{L}^p(X, \mu, \mathbb{K})$ for $p \in (0, \infty]$ is a vector space.

1. $\|\cdot\|_\infty : \mathcal{L}^\infty(X, \mu, \mathbb{K}) \to \mathbb{R}^+_0$ given by

   $$\|f\|_\infty := \inf\{\|g\|_{\sup} : g = f \text{ a.e. and } g : X \to \mathbb{K} \text{ bounded measurable}\}$$

   defines a seminorm on $\mathcal{L}^\infty(X, \mu, \mathbb{K})$, making it into a complete seminormed space.

2. If $1 \leq p < \infty$, then $\|\cdot\|_p : \mathcal{L}^p(X, \mu, \mathbb{K}) \to \mathbb{R}^+_0$ given by

   $$\|f\|_p := \left(\int_X |f|^p\right)^{1/p}$$

   defines a seminorm on $\mathcal{L}^p(X, \mu, \mathbb{K})$, making it into a complete seminormed space.

3. If $p \leq 1$, then $s_p : \mathcal{L}^p(X, \mu, \mathbb{K}) \to \mathbb{R}^+_0$ given by

   $$s_p(f) := \int_X |f|^p$$

   defines a semi-pseudo-norm on $\mathcal{L}^p(X, \mu, \mathbb{K})$, making it into a complete semi-pseudo-normed space.

Example 3.55. For any $p \in (0, \infty]$, the closure $N := \overline{\{0\}}$ of zero in $\mathcal{L}^p(X, \mu, \mathbb{K})$ is the set of measurable functions that vanish almost everywhere. The quotient space $\mathrm{L}^p(X, \mu, \mathbb{K}) := \mathcal{L}^p(X, \mu, \mathbb{K})/N$ is a complete mvs. It carries a norm (i.e., is a Banach space) for $p \geq 1$ and a pseudonorm otherwise. In the case $p = 2$ the norm comes from an inner product making the space into a Hilbert space.

3.7 The Banach-Steinhaus Theorem

Definition 3.56. Let S be a topological space. A subset $C \subseteq S$ is called *nowhere dense* iff its closure $\overline{C}$ does not contain any non-empty open set. A subset $C \subseteq S$ is called *meager* iff it is the countable union of nowhere dense subsets.

Proposition 3.57. *Let X and Y be tvs and $A \subseteq \mathrm{CL}(X, Y)$. Then A is (locally) equicontinuous iff for any neighborhood U of 0 in Y there exists a neighborhood V of 0 in X such that*

$$f(V) \subseteq W \quad \forall f \in A.$$

Proof. Immediate. □

Theorem 3.58 (Banach-Steinhaus)**.** *Let X and Y be tvs and $A \subseteq \mathrm{CL}(X, Y)$. For $x \in X$ define $A(x) := \{f(x) : f \in A\} \subseteq Y$. Define $B \subseteq X$ as*

$$B := \{x \in X : A(x) \text{ is bounded}\}.$$

If B is not meager in X, then $B = X$ and A is equicontinuous.

Proof. We suppose that B is not meager. Let U be an arbitrary neighborhood of 0 in Y. Choose a closed and balanced subneighborhood W of 0. Set

$$E := \bigcap_{f \in A} f^{-1}(W)$$

and note that E is closed and balanced, being an intersection of closed and balanced sets. If $x \in B$, then $A(x)$ is bounded, there exists $n \in \mathbb{N}$ such that $A(x) \subseteq nW$ and hence $x \in nE$. Therefore,

$$B \subseteq \bigcup_{n=1}^{\infty} nE.$$

If all sets nE were meager, their countable union would be meager and also the subset B. Since by assumption B is not meager, there must be at least one $n \in \mathbb{N}$ such that nE is not meager. But since the topology of X is scale invariant, this implies that E itself is not meager. Thus, the interior $\mathring{E} = \mathring{\overline{E}}$ is not empty. Also, $\mathring{E}$ is balanced since E is balanced and thus must contain 0. In particular, $\mathring{E}$, being open, is therefore a neighborhood of 0 and so is E itself. Thus,

$$f(E) \subseteq W \subseteq U \quad \forall f \in A.$$

This means that A is equicontinuous. Let now $x \in X$ arbitrary. Since x is bounded, there exists $\lambda > 0$ such that $x \in \lambda E$. But then, $f(x) \in f(\lambda E) \subseteq \lambda U$ for all $f \in A$. That is, $A(x) \subseteq \lambda U$, i.e., $A(x)$ is bounded and $x \in B$. Since x was arbitrary, $B = X$. □

Proposition 3.59. *Let S be a complete metric space and $C \subseteq S$ a meager subset. Then, C does not contain any non-empty open set. In particular, $C \neq S$.*

Proof. Since C is meager, there exists a sequence $\{C_n\}_{n\in\mathbb{N}}$ of nowhere dense subsets of S such that $C = \bigcup_{n\in\mathbb{N}} C_n$. Define $U_n := S \setminus \overline{C_n}$ for all $n \in \mathbb{N}$. Then, each U_n is open and dense in S. Thus, by Baire's Theorem 1.59 the intersection $\bigcap_{n\in\mathbb{N}} U_n$ is dense in S. Thus, its complement $\bigcup_{n\in\mathbb{N}} \overline{C_n}$ cannot contain any non-empty open set. The same is true for the subset $C \subseteq \bigcup_{n\in\mathbb{N}} \overline{C_n}$. □

Corollary 3.60. *Let X be a complete mvs, Y be a tvs and $A \subseteq \mathrm{CL}(X, Y)$. Suppose that $A(x) := \{f(x) : f \in A\} \subseteq Y$ is bounded for all $x \in X$. Then, A is equicontinuous.*

Proof. **Exercise.** □

Corollary 3.61. *Let X be a Banach space, Y a normed vector space and $A \subseteq \mathrm{CL}(X, Y)$. Suppose that*

$$\sup_{f\in A} \|f(x)\| < \infty \quad \forall x \in X.$$

Then, there exists $M > 0$ such that

$$\|f(x)\| < M\|x\| \quad \forall x \in X, \forall f \in A.$$

Proof. **Exercise.** □

3.8 The Open Mapping Theorem

Definition 3.62. Let S, T be topological spaces and $f : S \to T$. For $a \in S$ we say that f is *open at a* iff for every neighborhood U of a the image $f(U)$ is a neighborhood of $f(a)$. We say that f is *open* iff it is open at every $a \in S$.

Proposition 3.63. *Let S, T be topological spaces and $f : S \to T$. f is open iff it maps any open set to an open set.*

Proof. Straightforward. □

Theorem 3.64. *Let X be a Hausdorff tvs and C a closed vector subspace.*

1. *The quotient map $q : X \to X/C$ is linear, continuous and open. Moreover, the quotient topology on X/C is the only topology such that q is continuous and open.*

2. *The image of a base of the filter of neighborhoods of 0 in X is a base of the filter of neighborhoods of 0 in X/C.*

3. *The quotient tvs X/C inherits local convexity, local boundedness, metrizability, normability and completeness if X possesses the respective property.*

Proof. **Exercise.** □

Theorem 3.65 (Open Mapping Theorem)**.** *Let X be a complete mvs, Y a Hausdorff tvs, $f \in \mathrm{CL}(X, Y)$ and $f(X)$ not meager in Y. Then, Y is a complete mvs and f is open and surjective.*

Proof. Suppose U is a neighborhood of 0 in X. Let $V \subseteq U$ be a balanced subneighborhood of 0. Since every point of X is bounded we have

$$X = \bigcup_{n\in\mathbb{N}} nV \quad \text{and hence} \quad f(X) = \bigcup_{n\in\mathbb{N}} nf(V).$$

But $f(X)$ is not meager, so $nf(V)$ is not meager for at least one $n \in \mathbb{N}$. But then scale invariance of the topology of Y implies that $f(V)$ itself is not meager. Thus, $\overset{\circ}{\overline{f(V)}}$ is not empty, is open and balanced (since V is balanced) and thus forms a neighborhood of 0 in Y. Consequently, $\overline{f(V)}$ is also a neighborhood of 0 in Y and so is $\overline{f(U)}$.

Consider now a compatible pseudonorm on X. Let U be a neighborhood of 0 in X. There exists then $r > 0$ such that $B_r(0) \subseteq U$. Let $y_1 \in \overline{f(B_{r/2}(0))}$. We proceed to construct sequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ by induction. Supposed we are given $y_n \in \overline{f(B_{r/2^n}(0))}$. By the first part of the proof $\overline{f(B_{r/2^{n+1}}(0))}$ is a neighborhood of 0 in Y. Thus,

$$f(B_{r/2^n}(0)) \cap \left(y_n + \overline{f(B_{r/2^{n+1}}(0))}\right) \neq \emptyset.$$

In particular, we can choose $x_n \in B_{r/2^n}(0)$ such that

$$f(x_n) \in y_n + \overline{f(B_{r/2^{n+1}}(0))}.$$

Now set $y_{n+1} := y_n - f(x_n)$. Then, $y_{n+1} \in \overline{f(B_{r/2^{n+1}}(0))}$ as the latter is balanced.

Since in the pseudonorm $\|x_n\| < r/2^n$ for all $n \in \mathbb{N}$, the partial sums $\{\sum_{n=1}^m x_n\}_{m\in\mathbb{N}}$ form a Cauchy sequence. (Use the triangle inequality). Since X is complete, they converge to some $x \in X$ with $\|x\| < r$, i.e., $x \in B_r(0)$. On the other hand

$$f\left(\sum_{n=1}^m x_n\right) = \sum_{n=1}^m f(x_n) = \sum_{n=1}^m (y_n - y_{n+1}) = y_1 - y_{m+1}.$$

Since f is continuous the limit $m \to \infty$ exists and yields

$$f(x) = y_1 - y \quad \text{where} \quad y := \lim_{m\to\infty} y_m.$$

Note that our notation for the limit y implies uniqueness which indeed follows from the fact that Y is Hausdorff.

We proceed to show that $y = 0$. Suppose the contrary. Again using that Y is Hausdorff there exists a closed neighborhood C of 0 in Y that does not contain y. Its preimage $f^{-1}(C)$ is a neighborhood of 0 in X by continuity and must contain a ball $B_{r/2^n}(0)$ for some $n \in N$. But then $f(B_{r/2^n}(0)) \subseteq C$ and $\overline{f(B_{r/2^n}(0))} \subseteq C$ since C is closed. But $y_k \in \overline{f(B_{r/2^n}(0))} \subseteq C$ for all $k \geq n$. So no y_k for $k \geq n$ is contained in the open neighborhood $Y \setminus C$ of y, contradicting convergence of the sequence to y. We have thus established $f(x) = y_1$. But since $x \in B_r(0)$ and $y_1 \in \overline{f(B_{r/2}(0))}$ was arbitrary we may conclude that $\overline{f(B_{r/2}(0))} \subseteq f(B_r(0)) \subseteq f(U)$. By the first part of the proof $\overline{f(B_{r/2}(0))}$ is a neighborhood of 0 in Y. So we may conclude that $f(U)$ is also a neighborhood of 0 in Y. This establishes that f is open at 0 and hence open everywhere by linearity.

Since f is open the image $f(X)$ must be open in Y. On the other hand $f(X)$ is a vector subspace of Y. But the only open vector subspace of a tvs is the space itself. Hence, $f(X) = Y$, i.e., f is surjective.

Let now $C := \ker f$. Since f is surjective, Y is naturally isomorphic to the quotient space X/C as a vector space. Since f is continuous and open Y is also homeomorphic to X/C by Theorem 3.64.1 and hence isomorphic as a tvs. But then Theorem 3.64.3 implies that Y is metrizable and complete. □

Corollary 3.66. *Let X, Y be complete mvs and $f \in \mathrm{CL}(X,Y)$ surjective. Then, f is open.*

Proof. **Exercise.** □